FORMULAE FOR DERIVATIVES

Newton's forward difference interpolation formula is

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\,\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\,\Delta^3 y_0 + \ldots. \qquad (1)$$

where $$u = \frac{x-a}{h} \qquad (2)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\,\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\,\Delta^3 y_0 + \ldots \qquad (3)$$

Differentiating eqn. (2) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \qquad (4)$$

We know that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \ldots\right] \qquad (5)$$

Expression (5) provides the value of $\frac{ay}{dx}$ at any x which is not tabulated.

Formula (5) becomes simple for tabulated values of x, in particular when $x = a$ and $u = 0$

Putting $u = 0$ in (5), we get

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\,\Delta^2 y_0 + \frac{1}{3}\,\Delta^3 y_0 - \frac{1}{4}\,\Delta^4 y_0 + \frac{1}{5}\,\Delta^5 y_0 - \ldots\right]} \qquad (6)$$

Differentiating eqn. (5) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right] \quad (7)$$

Putting $u = 0$ in (7), we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\,\Delta^4 y_0 + \ldots\right) \quad (8)$$

Similarly, we get

$$\left(\frac{d^3 y}{dx^3}\right)_{x=a} = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\,\Delta^4 y_0 + \ldots\right) \quad (9)$$

and so on.

Formulae for computing higher derivatives may be obtained by successive differentiation.

Newton's backward difference interpolation formula is

$$y = y_n + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\,\nabla^3 y_n + \ldots$$

where $u = \dfrac{x - x_n}{h}$

Differentiating with respect to, u, we get

$$\frac{dy}{du} = \nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2 + 6u + 2}{6}\right)\nabla^3 y_n + \ldots$$

now differentiating with respect to, x, we get

$$\frac{du}{dx} = \frac{1}{h}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2+6u+2}{6}\right)\nabla^3 y_n + \dots\right]$$

At $x = x_n$, we have $u = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \dots\right)$$

Similarly, we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right)$$

$$\left(\frac{d^3 y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \dots\right)$$

and so on.

For unequally spaced values of the argument

(*i*) **Newton's divided difference formula is**

$$f(x) = f(x_0) + (x - x_0)\, \Delta f(x_0) + (x - x_0)(x - x_1)\, \Delta^2 f(x_0) + (x - x_0)(x - x_1)$$

$$(x - x_2)\, \Delta^3 f(x_0) + (x - x_0)(x - x_1)\,(x - x_2)(x - x_3)\, \Delta^4 f(x_0) + \dots$$

$f'(x)$ is given by

$$f'(x) = \Delta f(x_0) + \{2x - (x_0 + x_1)\}\, \Delta^2 f(x_0) + \{3x^2 - 2x(x_0 + x_1 + x_2)$$

$$+ (x_0 x_1 + x_1 x_2 + x_2 x_0)\}\, \Delta^3 f(x_0) + \dots$$

(*ii*) **Lagrange's interpolation formula is**

$$f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)} f(x_1) + \dots$$

$f'(x)$ can be obtained by differentiating $f(x)$

Example

From the following table of values of x and y, obtain $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *for x = 1.2, 2.2 and 1.6*

x:	*1.0*	*1.2*	*1.4*	*1.6*	*1.8*	*2.0*	*2.2*
y:	*2.7183*	*3.3201*	*4.0552*	*4.9530*	*6.0496*	*7.3891*	*9.0250.*